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Random walks on ultrametric spaces: mean and variance of the range

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Abstract. We study thermally activated hopping processes on regularly multifurcating ultrametric spaces (UMS) and investigate the first two moments of the range R_n, where n is the number of steps. Paralleling the recent findings for Cayley trees the knowledge of exact generating functions for these moments permits to draw conclusions about their behaviour for large n. For the first moment, the mean number of distinct sites visited, S_n, we derive analytically the cross-over from a linear behaviour in the transient regime $\gamma > 1$ to a sublinear increase $S_n \sim n^\gamma$ in the recurrent regime $\gamma < 1$. Here as usual for UMS, the parameter γ depends linearly on the temperature. The second moment of R_n, the variance V_n, displays different behaviours corresponding to strongly transient ($\gamma > 2$), not strongly transient ($1 < \gamma < 2$) and recurrent ($\gamma < 1$) random walks. All results are verified by comparison to computer data obtained through simulations and through numerical series expansions.

1. Introduction

During the last few years ultrametric spaces (UMS) [1-5] and related models [6-8] have attracted a lot of interest as a means to describe energetic disorder. The hierarchical arrangement of UMS in energy space complements a series of recursively generated models such as hierarchical arrangements in real space (such as fractals) and processes which involve hierarchic timescales (such as continuous time random walks CTRW) [9]. Furthermore UMS possess an additional advantage, in that the master equation which governs diffusion can be solved exactly [1, 2]. Thus UMS represent a highly non-trivial model of disorder which allows detailed and exact analytical studies of random walk (RW) properties.

An important quantity in RW theory is the *range* R_n which denotes the number of distinct sites visited in n steps. The range plays a crucial role in the trapping problem, where particles diffuse on lattices with randomly distributed traps and are captured on encounter [9, 10]. A completely analytical treatment of trapping is, in general, out of reach, because it requires the knowledge of the full R_n-distribution. Apart from simple one-dimensional models such information is lacking [10]. One generally resorts to simulations of the range [4, 10-14], to using limit theorems in the large n regime [15, 16], or to a study of the moments of R_n.

The fundamental works of Montroll and Weiss [17] stressed the generating function method in the study of the mean number of distinct sites visited, $S_n \equiv \langle R_n \rangle$. For RW-classification purposes the relation $S_n/n \to (1-F)$ for $n \to \infty$, where F is the probability to return ever again to the origin 0, is fundamental, see Spitzer [18]. RWs are called *transient* when $F < 1$, i.e. when there is a finite probability that the walker never

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returns to 0. Otherwise they are called *recurrent*. Most studies of transient and recurrent walks are carried out on d-dimensional regular Euclidean lattices [17, 19] For $d = 1$ and 2 simple symmetric RWs are recurrent, whereas for all $d > 2$ simple RW become transient.

Several extensions of RW schemes may be found in the review by Havlin and Ben-Avraham [20]. Thus one can envisage one-dimensional hopping models with a deterministic, hierarchical arrangement of barriers between next neighbours [6, 21] Here we will also endow the ultrametric structure with a deterministic, power-law distribution of transition rates. However, the coupling-scheme between distinct points of the UMS differs from the one-dimensional nearest-neighbour hopping models of references [6, 21], because in our case each site is directly connected to *all* sites of the UMS [2].

In this article we give a detailed study of S_n on UMS by using exact RW results [2, 5]. Following reference [5], we consider thermally activated processes, where the hopping rate ε_m between sites with ultrametric distance m is given by $\varepsilon_m \equiv (R/b)^m/\tau$. Here $R \equiv \exp(-\Delta/kT)$ is the Boltzmann factor, Δ the constant energy spacing between successive hierarchies of the tree, and τ the characteristic timescale. Much of the temporal behaviour can be described by the parameter $\gamma = (kT/\Delta) \ln b = \ln b/\ln(1/R)$, where 2γ plays the role of an effective spectral dimension $\tilde{d}$ [9, 22]. Thus, for instance, the probability $P_0(t)$ of being at the origin 0 after time t is given by $P_0(t) \sim t^{-\gamma}$. Now, ultrametric spaces provide the possibility to study RW under widely different conditions: RWs on UMS are recurrent for $bR < 1$ (i.e. $\gamma < 1$) and transient for $bR > 1$ (i.e $\gamma > 1$) [5] Moreover, for $bR^2 > 1$, (i.e. $\gamma > 2$), RWs are even *strongly transient*, a notion defined following Jain and Orey [23] letting f_k be the probability for first revisitation of the origin of the RW in the kth step, transient RWs are characterized by $F = \sum_{k=1}^{\infty} f_k < 1$. For strong transience one requires furthermore that $\sum_{n=0}^{\infty} r_n < \infty$, where r_n is the sequence of partial sums $r_n \equiv \sum_{k=n+1}^{\infty} f_k$ As we will show in the following, strong transience will turn out to be of great importance in the study of the *second* moment of R_n, the variance $V_n \equiv \langle R_n^2 \rangle - \langle R_n \rangle^2$. Examples for strongly transient RWs are simple, symmetric RWs on d-dimensional regular lattices with $d \geq 5$ [16], but also RWs on Cayley trees [24]. Interestingly, however, the UMS yield a model which can be swept continuously through all regimes—recurrent, transient and strongly transient—by varying the temperature-parameter γ. Thus S_n shows a cross-over behaviour from $S_n \sim n^{\gamma}$ in the recurrent to $S_n \sim n$ in the transient regime.

While S_n is a widely studied quantity in RW theory, much less is known about the higher moments of R_n. For the variance V_n extensive mathematical work provide the leading terms of V_n for large n [16, 19, 25] on regular lattices only. Recently we succeeded in establishing, using generating functions, an asymptotic expression for V_n on CT [24]. The method developed in [24] will be applied here to regularly multifurcating UMS. We obtain an exact expression for the leading term of V_n in the strongly transient regime of the RW. The same method provides upper bounds for the increase of the variance in the transient, but not strongly transient regime. In the recurrent regime it points to $V_n \sim Cn^{2\gamma}$, where the prefactor C shows an interesting oscillating behaviour for decreasing γ. All analytical results are verified by computer data. Hereby we make use of two methods: firstly, we obtain S_n and V_n from the whole distribution of R_n, which we generate through Monte Carlo simulations, paralleling [4] and [26]. As discussed in [26], one obtains a rather smooth R_n-distribution for $\gamma > 1$. For small γ, $\gamma < 1$, however, the distribution becomes very irregular, displaying families of sharp spikes, which render the numerical analyses delicate [26]. As a further method, we

obtain S_n and V_n from the numerical Taylor expansion of the corresponding generating functions determined as before. A comparison of the S_n and V_n produced by both methods shows excellent agreement at all γ values, confirming the good quality of the Monte Carlo simulations

2. The generating functions of the RW

Starting point of our analysis is the general solution of the master equation for arbitrary UMS which was derived by Bachas and Huberman [2]. For regularly multifurcating UMS with branching ratio b and L hierarchies of barriers one obtains [2]

$$P_0(t) = b^{-L} + a \sum_{k=1}^{L} b^{-k} \exp(\lambda_k t) \tag{1}$$

and

$$P_m(t) = b^{-L} + a \sum_{k=1}^{L} b^{-k} \exp(\lambda_k t) - a \sum_{k=1}^{m} b^{-k} \exp(\lambda_k t) - b^{-m} \exp(\lambda_m t) \tag{2}$$

Here $P_0(t)$ is the probability of being at the starting point (called origin) 0 at time t. Analogously, $P_m(t)$ is the probability to be in one of the ab^{m-1} points with ultrametric distance $m \geq 1$ from the origin at time t. Moreover, we set $a = b - 1$, and λ_k are the eigenvalues of the rate matrix. One can see that the difference between $P_0(t)$ and $P_m(t)$ consists only in a finite sum of correction terms. Moreover one has [5]

$$\lambda_k = -\frac{1}{\tau} \frac{a}{b} \frac{R}{1-R} \left(R^k \frac{b-R}{aR} - R^L \right). \tag{3}$$

We are interested in infinite systems $L \to \infty$, for which R_n increases continuously in the limit $n \to \infty$. Because of the homogeneity of the UMS (i.e. the equivalence of all sites), the number of steps performed up to time t obeys a Poisson process with waiting-time distribution $\psi(t) = \lambda \exp(-\lambda t)$, where the rate of leaving a site is $\lambda = aR(1-R^L)/[\tau b(1-R)]$. Thus it is possible to switch from the Laplace transforms $\hat{P}_m(u)$ to the generating functions $\tilde{P}_m(z)$ of the walk as exemplified in [5]. This is due to the fact that both are connected via $\hat{P}_m(u) = [1 - \hat{\psi}(u)] \tilde{P}_m[z = \hat{\psi}(u)]/u$, and since, defining $z \equiv \hat{\psi}(u)$, one has $u = \lambda(1-z)/z$. Paralleling [5] we have (where we set $\xi_k = \lambda_k/\lambda < 0$)

$$\tilde{P}_0(z) = \frac{b^{-L}}{1-z} + a \sum_{k=1}^{L} b^{-k} \frac{1}{1 - z - z\xi_k} \tag{4}$$

and

$$\tilde{P}_m(z) = \frac{b^{-L}}{1-z} + a \sum_{k=1}^{L} b^{-k} \frac{1}{1 - z - z\xi_k} - a \sum_{k=1}^{m} b^{-k} \frac{1}{1 - z - z\xi_k} - b^{-m} \frac{1}{1 - z - z\xi_m}. \tag{5}$$

Furthermore, we obtain from the identity $\tilde{S}(z) = [(1-z)^2 \tilde{P}_0(z)]^{-1}$, which is valid for arbitrary homogeneous lattices [5, 17], the generating function of the S_n

$$\tilde{S}(z) = \frac{1}{(1-z)^2} \left(\frac{b^{-L}}{1-z} + a \sum_{k=1}^{L} b^{-k} \frac{1}{1 - z - z\xi_k} \right)^{-1}. \tag{6}$$

3. S_n in the strongly transient regime ($\gamma > 2$)

In order to extract the leading terms of S_n from its generating function, we proceed as in [24]. first we take the limit $L \to \infty$, and expand $\tilde{S}(z)$ in terms of small positive $x = 1 - z$, i.e. for z closely below 1. Writing down the first two order terms for an expansion $x \to 0^+$ yields

$$\tilde{S}(x) \sim \frac{1}{x^2} \frac{(b-R)(bR-1)}{a^2 R} + \frac{1}{x} \frac{b(1-R)^2(1+R)(bR-1)}{a^2 R(bR^2-1)}. \tag{7}$$

From the small-x behaviour we extract the two leading terms of S_n by Tauberian arguments: we notice that the generating functions of the 'pure n-dependences' read $\Sigma_{n=0}^{\infty}(1-x)^n = 1/x$, $\Sigma_{n=0}^{\infty} n(1-x)^n = 1/x^2 - 1/x$, $\Sigma_{n=0}^{\infty} n^2(1-x)^n = 2/x^3 - 3/x^2 + 1/x$, and in general, $\Sigma_{n=0}^{\infty} n^{\alpha}(1-x)^n \sim \Gamma(\alpha+1)/x^{\alpha+1}$ for $x \to 0$ and for all $\alpha > -1$. Comparing this to equation (7), one obtains in the n-regime

$$S_n \sim \frac{(b-R)(bR-1)}{a^2 R} n + \frac{(bR-1)^2}{a(bR^2-1)}. \tag{8}$$

The first term $S_n^{(1)} = (b-R)(bR-1)n/(a^2R)$ yields the linear increase of S_n in the whole transient case $bR > 1$; the constant second term $S_n^{(2)}$, however, is valid only in the regime of strong transience and diverges for $bR^2 \to 1^+$ On the other hand, for infinitely high temperatures $R = 1$ the RW is allowed to reach any of the infinitely many lattice sites with equal probability. Since the system is infinitely large, $L \to \infty$, the walk visits at each jump a new site with probability one. Thus $S_n \sim n + 1$ holds, as can correctly be inferred from equation (8).

In order to check our analytical findings (especially the validity of the second-order terms in the asymptotic expansions, in view of possibly oscillating contributions [27]), we used two methods. First we performed Monte Carlo simulations on UMS for several branching ratios (mostly $b = 3$). To obtain a good distribution for R_n, we performed 10^4 distinct realizations of the walk and followed each walk for 10^4 steps.

Secondly, since we deal with exactly-known generating functions, equation (6), we evaluated the S_n numerically by expanding the Taylor series in n following the procedure described in [28]: one starts by noticing that the generating variable z in equation (6) appears only in the form of the geometric series $(1-cz)^{-1} = \Sigma_{n=0}^{\infty}(cz)^n$. Then a series $\Sigma_{n=0}^{\infty} \alpha_n z^n$ is readily inverted to the form $\Sigma_{m=0}^{\infty} \beta_m z^m \equiv (\Sigma_{n=0}^{\infty} \alpha_n z^n)^{-1}$ by using the identity

$$1 = \left(\sum_{n=0}^{\infty} \alpha_n z^n \right) \left(\sum_{m=0}^{\infty} \beta_m z^m \right) = \sum_{k=0}^{\infty} z^k \left(\sum_{n=0}^{k} \beta_n \alpha_{k-n} \right) \tag{9}$$

from which term-by-term comparison gives

$$\begin{aligned} &\alpha_0 \beta_0 = 1 \\ &\alpha_0 \beta_1 + \alpha_1 \beta_0 = 0 \end{aligned} \tag{10}$$

and generally $\Sigma_{n=0}^{k} \beta_n \alpha_{k-n}$ for all $k \geq 1$. Thus it is possible to calculate all the β_n iteratively from equation (10). Since equation (6) is given as a sum over k we chose L (i.e. the size of the system) quite large by taking $L = 80$. Here we employ this procedure to calculate S_n from equation (6), an analogous procedure will be applied to the variance in sections 6 to 8. As is shown in figure 1, both the simulation and the numerical Taylor expansion of equation (6) lead to S_n-values which agree very well to the analytical results.

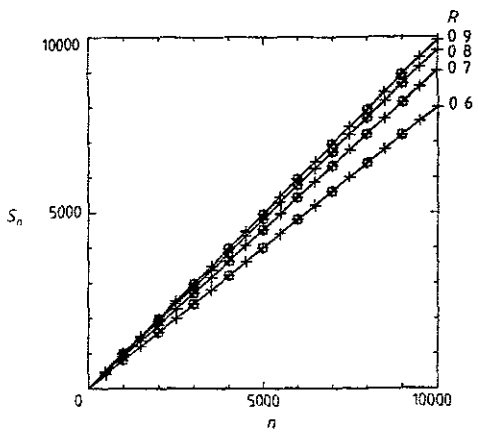

Figure 1. S_n plotted as a function of the number of steps n in the strongly transient regime ($\gamma > 2$) for $b = 3$ and for several values of R. The squares are the Monte Carlo results, the crosses refer to the numerical Taylor expansion, and the full lines are the analytical values obtained from equation (8).

Subtracting the first two terms of S_n, equation (8), from the numerical data yields indications about the *third* term $S_n^{(3)}$ of the expansion of S_n. The data point to an algebraic dependence $S_n^{(3)} \sim n^{2-\gamma}$ (notice that we are in the regime $\gamma > 2$!) at least for γ near 2. On the other hand, with increasing γ the term $S_n^{(3)}$ becomes very small and gets hidden inside the numerical resolution.

4. S_n in the transient, but not strongly transient regime ($1 < \gamma < 2$)

Now we turn to lower temperatures $bR^2 < 1$ (i.e. $1 < \gamma < 2$) and leave the *strongly* transient, but not the transient regime. In this case $\tilde{S}(x)$ is analysed paralleling the derivation given above, and we obtain

$$\tilde{S}(x) \sim \frac{1}{x^2} \frac{(b-R)(bR-1)}{a^2 R} + \frac{A}{x^{3-\gamma}} \tag{11}$$

where A is given by

$$A = \pi \frac{(bR-1)^2}{a \ln(1/R)} C^{\gamma-2} \operatorname{cosec}[\pi(\gamma-1)] \tag{12}$$

and the constant C is

$$C = aR/(b-R). \tag{13}$$

The form of A is due to the fact that sums of the type $\Sigma_{k=1}^{\infty} [(bR)^k (R^k + \varepsilon)]^{-1}$, which arise in the study of $\tilde{S}(x)$, can be replaced in the limit $\varepsilon \to 0$ by integrals, and by making use of equation (3.311.9) of [29]. Recast as a function of the number of steps, equation (11) yields:

$$S_n \sim \frac{(b-R)(bR-1)}{a^2 R} n + \frac{A}{\Gamma(3-\gamma)} n^{2-\gamma}. \tag{14}$$

As one can see, the leading terms in equations (8) and (14) are equal; this guarantees a linear increase of S_n with n, as we expect for a transient behaviour. However, the second term in equation (8), which was independent of n, shows now a sublinear algebraic increase with n, $S_n^{(2)} = An^{2-\gamma}/\Gamma(3-\gamma)$. Thus UMS parallel regular d-dimensional lattices where one has [19] in the strongly transient regime ($d \geq 5$) $S_n^{(2)} \sim$ const, whereas in the transient, but no strongly transient regime $S_n^{(2)}$ grows with n, namely as $O(n^{1/2})$ for $d = 3$ and as $O(\log n)$ for $d = 4$.

Figure 2 shows a log-log-plot of the numerical data. The squares denote the simulation results for S_n and the dots the values obtained through the Taylor expansion of equation (6). Both agree nicely to the analytical form of equation (14): the triangles are obtained by subtracting the first term $S_n^{(1)}$ from the numerical data. Analogously, the crosses denote $S_n - S_n^{(1)} - S_n^{(2)}$. One clearly remarks the slower increase of the remaining rest, verifying the asymptotic form $S_n \sim S_n^{(1)} + S_n^{(2)}$.

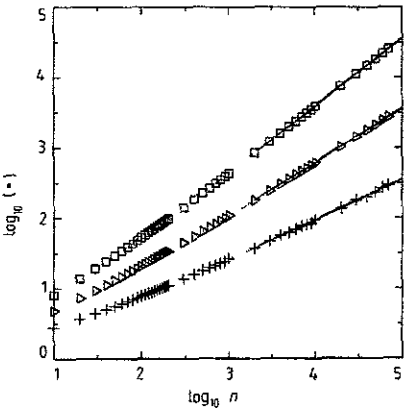

Figure 2. Data for S_n in the transient, but not strongly transient regime for $b = 3$ and $R = 0.4$ ($\gamma = 1.199$) The dots denote the numerical Taylor expansion The squares are the simulation results The triangles denote $S_n - S_n^{(1)}$, and the crosses $S_n - S_n^{(1)} - S_n^{(2)}$, where $S_n^{(1)}$ and $S_n^{(2)}$ are given by equation (14).

5. S_n in the recurrent regime ($0 < \gamma < 1$)

Now we leave the transient regime and turn to recurrent RW on UMS for which γ lies between 0 and 1. The calculations are carried out analogously to the transient case, and we find

$$\tilde{S}(x) \sim Bx^{-1-\gamma} \tag{15}$$

with

$$B = \ln(1/R)C^{-\gamma}/[a\pi \operatorname{cosec}(\pi\gamma)]. \tag{16}$$

Since we have to estimate sums in equation (15) by integrals (similar to the case of $S_n^{(2)}$ in section 3), we find here only the first leading term in x. Reformulated in terms of n, we obtain

$$S_n \sim Bn^{\gamma}/\Gamma(1+\gamma) \tag{17}$$

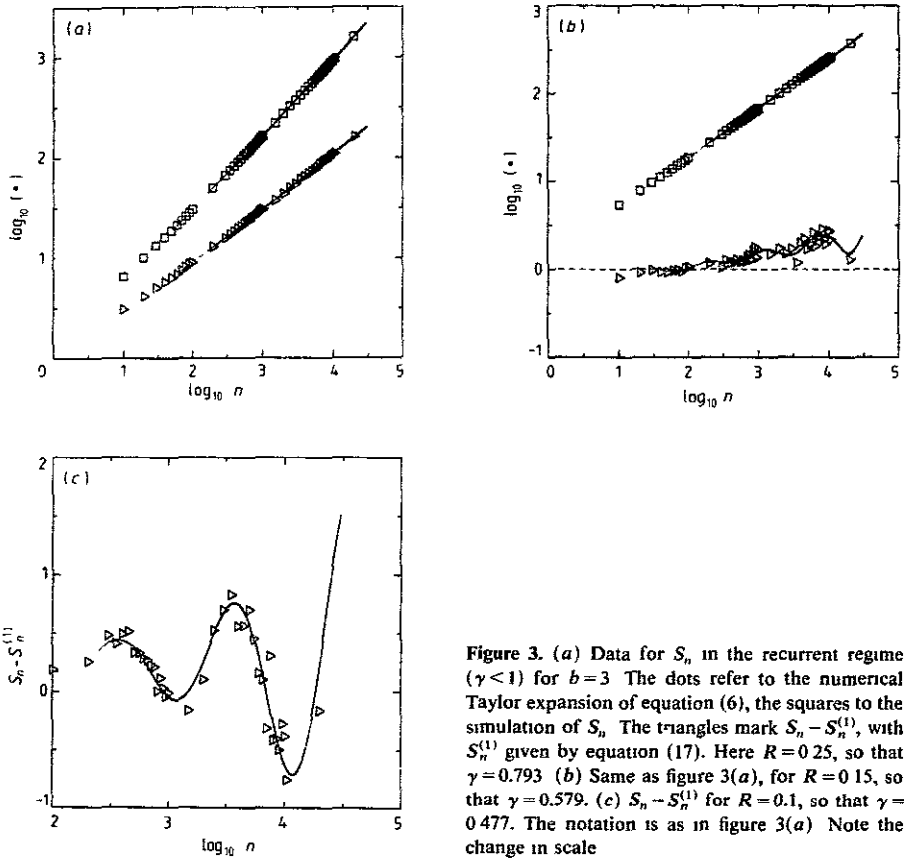

Figure 3. (*a*) Data for S_n in the recurrent regime ($\gamma < 1$) for $b = 3$. The dots refer to the numerical Taylor expansion of equation (6), the squares to the simulation of S_n. The triangles mark $S_n - S_n^{(1)}$, with $S_n^{(1)}$ given by equation (17). Here $R = 0.25$, so that $\gamma = 0.793$. (*b*) Same as figure 3(*a*), for $R = 0.15$, so that $\gamma = 0.579$. (*c*) $S_n - S_n^{(1)}$ for $R = 0.1$, so that $\gamma = 0.477$. The notation is as in figure 3(*a*). Note the change in scale.

i.e. a sublinear increase with n, $S_n \sim n^{\gamma}$. We note that the same cross-over from a linear increase of S_n for $\gamma > 1$ to a sublinear increase for $\gamma < 1$ was found in [5] by considering Markovian CTRW. Again, the leading term $S_n^{(1)}$ in equation (17) is supported very well by the numerical data. Interestingly $S_n - S_n^{(1)}$ used to estimate $S_n^{(2)}$ displays a drastic change well-inside the recurrent domain: for γ not much less than unity, the behaviour is reminiscent of the transient case, since both S_n and $S_n - S_n^{(1)}$ increase with n. Figure 3(*a*) shows the situation for $\gamma = 0.793$ ($b = 3$, $R = 0.25$). But around $\gamma = 0.5$ $S_n - S_n^{(1)}$ tends to a mean constant value and, with decreasing γ, starts to oscillate: figures 3(*b*) and 3(*c*) show the results for $\gamma = 0.579$ and $\gamma = 0.477$. We will obtain a similar behaviour for the variance in section 8.

6. Variance in the strongly transient regime

After having been able to describe the asymptotic behaviour of S_n in the three regimes (RW recurrent, transient and strongly transient), we turn now to the calculation of the variance V_n. This is considerably more involved than the determination of S_n, as, for instance, show the elaborate mathematical works for RW on regular Euclidean lattices

[16, 19, 23, 25]. The knowledge of V_n is quite valuable: for instance, it allows to determine, together with S_n, in the trapping problem the first two terms of the decay law [4, 11, 24] As we showed in [24], the following relation holds

$$V_n = Q_n + S_n - S_n^2 \tag{18}$$

where the Q_n are given in terms of their generating function $\tilde{Q}(z) = \sum_{n=0}^{\infty} z^n Q_n$ by

$$\tilde{Q}(z) = 2\tilde{S}(z)\tilde{H}(z) \tag{19}$$

with

$$\tilde{H}(z) = \sum_{i \neq 0} P(0, y, z)[P(0, 0; z) + P(0, y, z)]^{-1}. \tag{20}$$

Here $P(0, y, z)$ is the generating function of $P_n(0, y)$, the probability of being at the site y after n steps when the RW started at the origin 0. Equation (20) holds exactly for symmetric RW on homogeneous spaces. Remembering that there are ab^{m-1} sites y with ultrametric distance m from the origin, equation (20) can be rewritten as

$$\tilde{H}(z) = \frac{a}{b} \sum_{m=1}^{L} b^m \tilde{P}_m(z)[\tilde{P}_0(z) + \tilde{P}_m(z)]^{-1} \tag{21}$$

where $\tilde{P}_0(z)$ and $\tilde{P}_m(z)$ are given through equations (4) and (5).

Since all generating functions in equation (19) and equation (21) are available, we can apply the same methods as before to extract the behaviour of V_n for large n. For the analysis of $\tilde{H}(z = 1 - x)$, equation (21), we proceed similarly to the study of $\tilde{S}(x)$. Note, however, that we must take care to perform the various limits in the right order. Taking first the 'thermodynamic limit' $L \to \infty$ in the arguments of equation (20) would imply the neglect of the term $b^{-L}/(1-z)$. This term, together with the sum $\sum_{m=1}^{L} b^m$ provides, however, a finite contribution to the leading term in $\tilde{H}(x)$. Care must thus be applied in following the proper limiting procedures.

In the strongly transient case the analysis of $\tilde{H}(x)$ yields the following two contributions:

$$\tilde{H}(x) = \frac{1}{x} \frac{(b-R)(bR-1)}{a^2 R} + \frac{b(1-R)^2(1+R)(bR-1)}{a^2 R(bR^2-1)} - [1 + (1-R)\tau(b, R)] + o(1) \tag{22}$$

where the convergent series $\tau(b, R)$ is defined by:

$$\tau(b, R) = \sum_{m=1}^{\infty} (bR^2)^{-m}/[(bR)^{-m} + a/(b - bR)]. \tag{23}$$

The rest $o(1)$ in equation (22) diminishes for $x \to 0$ and plays no role in the asymptotic regime.

Together with equation (19), from equations (7) and (22) we can now obtain the asymptotic expansion for $\tilde{Q}(x) = 2\tilde{S}(x)\tilde{H}(x)$. From this we extract the asymptotic form of Q_n by using the Tauberian terms of section 3. Combining equations (8) and (18), the first term, which is proportional n^2, drops out, and the leading term of V_n turns out to be linear in n

$$V_n \sim n \frac{(b-R)(bR-1)(1-R)[b(1-R)(bR^2-1+2aR) - 2a^2(bR^2-1)R\tau(b, R)]}{a^4 R^2(bR^2-1)}. \tag{24}$$

Equation (24) agrees with a general result by Jain and Orey [23], who found for strongly transient RW a linear increase of the variance: $V_n \sim \text{const}\, n$. As can readily be seen in equation (24), the coefficient multiplying n diverges in the limit $bR^2 \to 1^+$, i.e. V_n becomes broader for decreasing temperatures. On the other hand, for infinitely high temperatures, $R = 1$, the variance vanishes. This agrees with the previous finding that for $R = 1$ one has $S_n = n + 1$; this implies, since $n+1$ is the maximum value for R_n, that the R_n-distribution is very strongly peaked at $n+1$.

Figure 4 shows the comparison of equation (24) to the Monte Carlo simulation and to the data which one obtains from equation (18) by performing an analytical Taylor expansion of $\tilde{Q}(z)$, equation (19), and $\tilde{S}(z)$, equation (6). As one can see, the agreement is very good. The difference term obtained by subtracting equation (24) from the data of the numerical Taylor expansions tends toward a constant, but the numerical accuracy for γ near 2 does not allow a precise statement.

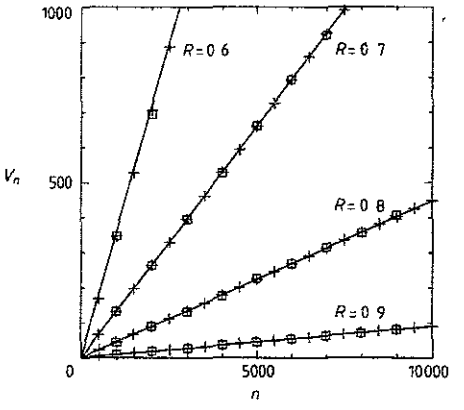

Figure 4. The variance V_n as a function of n. The parameters are as in figure 1, and the full lines are the analytical expression, equation (24).

7. Variance in the transient, but not strongly transient regime

On regular Euclidean lattices the dependence of V_n on n is in general nonlinear outside the strongly transient regime. We proceed to show that a similar behaviour holds for RW on UMS: we apply the same method as in the last section to the transient, but not strongly transient regime and obtain

$$\tilde{H}(x) = \frac{1}{x}\frac{(b-R)(bR-1)}{a^2 R} + \frac{(2-\gamma)A}{x^{2-\gamma}} \tag{25}$$

which implies:

$$V_n \sim \text{const}\, n^{3-\gamma}. \tag{26}$$

However, combining Q_n and S_n in equation (26), not only the first, but also the second term drops out, i.e. the constant prefactor of the leading term in equation (26) becomes zero. Thus the formalism yields only an upper bound for the variance, from which we

infer that V_n increases more slowly than $n^{3-\gamma}$. It is thus necessary to expand both $\tilde{S}(x)$ and $\tilde{H}(x)$ at least up to *third* order to extract the first non-vanishing term of V_n, an analytical procedure which is extremely cumbersome. Furthermore, we do not expect the law for V_n to be as simple as before: numerical data point to an algebraic increase $V_n \sim n^{\delta}$ with $1 < \delta < 3 - \gamma$, but δ does not appear to depend linearly on γ. Some of the results are shown in figure 5. The slope δ is extracted from fits to the numerical Taylor expansion of the generating functions for values of n up to $n = 20\,000$. We have chosen branching ratios b from 3 to 6 and find that all data points lie on the same curve. This shows that the exponent δ depends only on γ. The asymptotic regime of V_n is reached very slowly for $\gamma \approx 1$. For $\gamma \to 1$ we expect $\delta \to 2$ as a continuous connection to the behaviour of V_n in the recurrent regime.

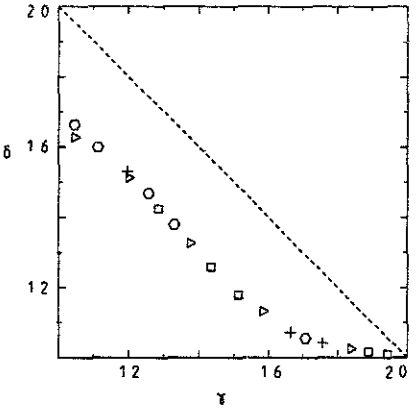

Figure 5. Data for the exponent δ of the variance $V_n \sim n^{\delta}$ in the transient, but not strongly transient regime ($1 < \gamma < 2$) The values are extracted from the numerical Taylor expansion for $b = 3$ (triangles), $b = 4$ (squares), $b = 5$ (crosses) and $b = 6$ (hexagons) The dashed line gives the upper bound, equation (26)

8. V_n in the recurrent regime

In the recurrent regime $0 < \gamma < 1$ the shape of the R_n-distribution changes drastically when compared to the situation for transient RW. As Monte Carlo simulations show [26], for $\gamma > 1$ the distribution of R_n is quite smooth and seems, for $\gamma \gg 1$, to tend to a Gaussian. For decreasing temperatures, however, the distribution looses its regularity and series of b-fold sharp spikes appear. Nevertheless, as we will show, the simulation results and the numerical Taylor expansion for V_n agree very well, which shows that the simulation data are correct and not mere artifacts.

Let us first consider the analytical evaluation of V_n. Here it turns out that the first terms in the expansion of S_n and Q_n do *not* cancel out (distinct from the situation discussed in section 7). Therefore the *first* expansion terms both of $\tilde{S}(x)$, given in equation (15), and of $\tilde{H}(x)$, equation (21), suffice to extract the leading term of V_n. We have analysed $\tilde{H}(x)$ and find that it follows the form

$$\tilde{H}(x) \sim D(x) x^{-\gamma} / B \tag{27}$$

where B is as in equation (16). Unlike the previous sections, however, the leading term $x^{-\gamma}$ is modulated by a prefactor $D(x)$ which displays an oscillatory behaviour for $x \to 0$ between two (positive) values: $C_1 \leqslant D(x) \leqslant C_2$. The exact analytic form of $D(x)$ is given by

$$D(x) = (1-R) \sum_{i=1-\nu}^{\infty} R^{i-\varepsilon}\{(1+R^{i-\varepsilon})(1+R^{i+1-\varepsilon}) \times [2B^{-1}-(1-R)\kappa_{i-1}][2B^{-1}-(1-R)\kappa_i]\}^{-1}. \tag{28}$$

In equation (28) ν is defined as being the largest integer below $\ln[C\epsilon/(1-x)]/\ln R$, where the constant C is given by equation (13). Moreover, ε is just the difference: $\varepsilon \equiv \ln[Cx/(1-x)]/\ln R - \nu$ and hence lies between 0 and 1. Finally, κ_k is defined as:

$$\kappa_k = [C/(1-x)]^{\gamma} \sum_{i=1-\nu}^{k} b^{-(i-\varepsilon)} R^{i-\varepsilon}/[(1+R^{i-\varepsilon})(1+R^{i+1-\varepsilon})]. \tag{29}$$

In equation (28), x influences the value of $D(x)$ in two ways. Firstly, for $x \to 0$ the lower limit $1-\nu$ of the sum tends to $-\infty$. However, this influence in equation (28) is negligible, since (except for $\gamma \approx 1$) the main contributions to the sum arise from i around 0. Of more importance is, secondly, the value of ε which fluctuates between

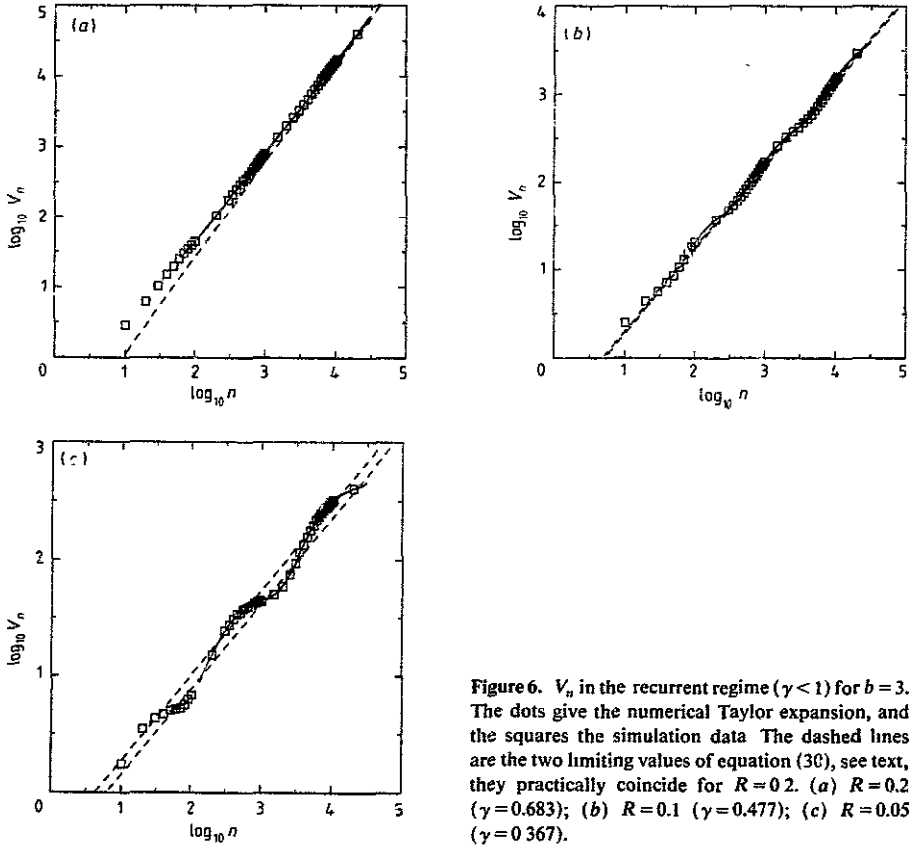

Figure 6. V_n in the recurrent regime ($\gamma < 1$) for $b = 3$. The dots give the numerical Taylor expansion, and the squares the simulation data The dashed lines are the two limiting values of equation (30), see text, they practically coincide for $R = 0.2$. (*a*) $R = 0.2$ ($\gamma = 0.683$); (*b*) $R = 0.1$ ($\gamma = 0.477$); (*c*) $R = 0.05$ ($\gamma = 0.367$).

0 and 1 when x tends to 0. As a numerical evaluation of equation (28) shows, this causes an oscillation of the value of $D(x)$ with an increasing amplitude when R gets small. It is possible to calculate numerically both extreme values C_1 and C_2 of $D(x)$ by varying ε within the unit interval Now we obtain in the n-regime for the variance

$$V_n \sim n^{2\gamma}[2\bar{D}/\Gamma(1+2\gamma) - B^2/\Gamma(1+\gamma)^2] \tag{30}$$

where we considered $\bar{D}$ to be a constant lying between C_1 and C_2, depending on the special choice of ε. We can therefore find two limiting laws for the variance and expect that the true behaviour of V_n displays oscillations between them As figure 6 shows, equation (30) is supported well by the numerical data. An increase in V_n proportional to $n^{2\gamma}$ is clearly visible, but this form is reached slowly in the region where γ is below but close to 1. In this region $\bar{D}$ is nearly independent of the special choice of ε, and the two dashed lines which denote the two limiting cases practically coincide. For smaller γ values the algebraic law $V_n \sim n^{2\gamma}$ continues to hold, but (as for the difference $S_n - S_n^{(1)}$ discussed previously) V_n starts to oscillate with increasing amplitude for decreasing γ. The two extreme values of $D(x)$ fit quite well to the amplitude of the oscillations, and thus we expect that for $n \to \infty$, V_n will lie well within the region bounded by the two dashed lines.

Hence for recurrent RW *we established (in the range of parameters investigated)* that $V_n \sim S_n^2$, a result which holds exactly in one dimension and which was also found numerically for RW over compact spaces [4, 12].

9. Conclusions

Summarizing the results, we have analysed the first two moments of the range R_n, the mean S_n and the variance V_n, for thermally activated random walks on ultrametric spaces, in order to gain a better understanding of the dynamics on systems without translational symmetry. Using methods recently employed by us for Cayley trees, the knowledge of the exact generating functions allowed us to determine the two leading terms of S_n for transient RW, i.e. for $\gamma > 1$, and the leading term of S_n in the recurrent regime $0 < \gamma < 1$. Here one finds a characteristic cross-over from a linear increase of S_n for high γ values to $S_n \sim n^{\gamma}$ for small γ values, $\gamma < 1$. For the leading term of the variance V_n we found in the strongly transient regime ($\gamma > 2$) an exact expression; $V_n^{(1)}$ increases linearly with n, a fact similar to the situation on highly dimensional regular lattices and on Cayley trees, and in agreement with the general theory for strongly transient RW. In the transient, but not strongly transient regime $1 < \gamma < 2$ our investigation leads to an upper bound to V_n which depends on the parameter γ. In the recurrent case the analysis displays an algebraic increase of the variance, V_n behaving mainly as $n^{2\gamma}$, with superimposed oscillations. All our analytical results agree very well with the Monte Carlo simulations and with the data obtained from the numerical Taylor expansion of the generating functions. These results may be of interest in the target problem, the trapping problem, or diffusion-limited reactions on UMS [9, 20].

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